DIMENSION-REDUCTION SIMULATION OF STATIONARY STOCHASTIC VECTOR PROCESSES VIA THE VAR MODEL

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Abstract

This study proposes an enhanced dimension-reduction (DR) method based on vector auto-regression (VAR) models to address the dimension dilemma inherent in conventional models caused by their reliance on high-dimensional random variables. The theoretical framework for AR and VAR models is derived based on spectral decomposition theory for stationary stochastic processes, revealing that the randomness of VAR models is inherently determined by high-dimensional random variables, which causes conventional models relying on Monte Carlo (MC) simulations to inevitably encounter the dimension dilemma. To circumvent this, a DR method based on random orthogonal functions is proposed, where high-dimensional random variables can be represented by just two elementary random variables. The number-theoretic method is applied to select a representative point set with complete probabilistic information, thereby efficiently generating a representative sample set of the stationary vector process with full probabilistic information. For illustrative purposes, a stationary stochastic wind speed vector process is generated, and the parameter sensitivity analysis of the DR-VAR method is also discussed. Numerical results show that the proposed method has significant advantages in both accuracy and efficiency compared to the conventional VAR model. Furthermore, the DR-VAR method is a full probabilistic approach, which is essentially consistent with the Probability Density Evolution Method (PDEM) within the framework of the third-generation structural theory. Consequently, the DR-VAR method can be further integrated with PDEM, laying a solid foundation for rigorous stochastic wind-induced response and wind resistance reliability analyses of complex engineering structures.

Keywords*: vector auto-regressive model; dimension-reduction representation; random orthogonal functions; stationary stochastic vector process*

1. Introduction

Wind load is the governing load for large-scale engineering structures such as long-span bridges, high-voltage transmission towers, and large wind turbines, and is characterized by pronounced randomness and significant spatial-temporal variability [1]. In engineering practice, wind loads are commonly modeled as stationary stochastic vector processes, and their accurate simulation is essential for wind-induced dynamic response analysis and wind resistance reliability assessment [2]. Current methods for stochastic wind field simulation can be broadly classified into two categories [3]: the frequency-domain spectral decomposition methods and the time-domain filtering methods. The frequency-domain methods require matrix decomposition at each discrete frequency point, resulting in substantial computational cost and memory consumption. In contrast, time-domain methods, such as the VAR model, exploit the autoregressive relationship of stochastic signals to generate sample ensembles without requiring frequency-domain decomposition, thus offering superior computational efficiency. However, the conventional VAR model relies on Monte Carlo sampling to handle high-dimensional random variables, making it difficult to obtain representative sample sets with complete probabilistic information [4]. To this end, this paper introduces the dimension-reduction representation based on random orthogonal functions into the VAR model, and proposes a dimension-reduction VAR (DR-VAR) method. In this method, the high-dimensional random variables are expressed as functions of merely two elementary random variables, and representative points are selected via the number-theoretic method, enabling the generation of a representative sample ensemble of stationary stochastic vector processes with complete probabilistic information and thereby providing a foundation for stochastic dynamic response and reliability analyses at the probability density level.

2. AR model of univariate stationary stochastic processes

Let $V(t)$ be a zero-mean real-valued stationary stochastic process, which can be expressed in the following Fourier-Stieltjes integral form [5]:

$$V(t)=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}Z(\omega)=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}H(\omega)\mathrm{d}U(\omega) \tag{1}$$

where $\mathrm{i}=\sqrt{-1}$ is the imaginary unit; ω denotes the circular frequency; $Z(\omega)$ and $U(\omega)$ are orthogonal increment processes, and the increments of these processes, i.e., $\mathrm{d}Z(\omega)$ and $\mathrm{d}U(\omega)$, satisfy the following conditions [6]:

$$E\left[\mathrm{d}Z(\omega)\right]=E\left[\mathrm{d}U(\omega)\right]=0\,,\,\mathrm{d}Z(\omega)=\mathrm{d}Z^*(-\omega)\,,\,\mathrm{d}U(\omega)=\mathrm{d}U^*(-\omega) \tag{2a}$$

$$E\left[\mathrm{d}Z(\omega)\mathrm{d}Z^*(\omega')\right]=S_V(\omega)\delta_{\omega\omega'}\mathrm{d}\omega\,,\,E\left[\mathrm{d}U(\omega)\mathrm{d}U^*(\omega')\right]=\delta_{\omega\omega'}\mathrm{d}\omega \tag{2b}$$

where $E[\cdot]$ denotes the mathematical expectation; superscript "$*$" denotes the complex conjugate; $\delta_{\omega\omega'}$ denotes the Kronecker delta, i.e., $\delta_{\omega\omega'}=1$ for $\omega=\omega'$ and $\delta_{\omega\omega'}=0$ for $\omega\neq\omega'$; $S_V(\omega)$ is the power spectral density (PSD) function of the stationary stochastic process $V(t)$.

In Eq. (1), $H(\omega)$ is the frequency response function which satisfies the following condition:

$$S_V(\omega)=\left|H(\omega)\right|^2 \tag{3}$$

Further, define $h(t)$ as the inverse Fourier transform of $H(\omega)$:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega \tag{4}$$

Then, the stationary stochastic process $V(t)$ defined in Eq. (1) can be rewritten in the time-domain form as follows:

$$V(t) = \int_{-\infty}^{\infty} h(t-\tau)W(\tau)\mathrm{d}\tau \tag{5}$$

where $W(t)$ is a white noise process, which satisfies the following conditions:

$$E\left[W(t)W^*(t')\right] = 2\pi\delta(t-t'), \quad S_W(\omega) = 1 \tag{6}$$

where $\delta(t-t')$ denotes the Dirac delta function; $S_W(\omega)$ is the PSD function of white noise process $W(t)$.

Noting that $V(t)=0$ for $t<0$, and $h(t-\tau)=0$ for $t<\tau$, Eq. (5) can thus be rewritten as:

$$V(t) = \int_0^t h(t-\tau)W(\tau)\mathrm{d}\tau = \int_0^{t-(q+1)\Delta t} h(t-\tau)W(\tau)\mathrm{d}\tau + \int_{t-(q+1)\Delta t}^{t} h(t-\tau)W(\tau)\mathrm{d}\tau \tag{7}$$

where Δt denotes the sampling time interval; q is an integer.

For each integral time interval $\left[t-(i+1)\Delta t, t-i\Delta t\right], (i=0,1,\cdots,q)$, approximating $h(t-\tau)$ as the constant $h(i\Delta t)$ and letting $Y_{t-i} = \int_{t-(i+1)\Delta t}^{t-i\Delta t} W(\tau)\mathrm{d}\tau$, Eq. (7) is simplified to:

$$V_t = \sum_{i=0}^{q} h(i\Delta t)Y_{t-i} \tag{8}$$

where Y_{t-i} are zero-mean uncorrelated random variables obtained through white noise integration, satisfying the following conditions:

$$E\left[Y_{t-i}\right] = 0, \quad E\left[Y_{t-i}Y_{t-j}\right] = 2\pi\Delta t\delta_{ij} \tag{9}$$

Further, let $\theta_i = h(i\Delta t)/h(0)$, $X_{t-i} = h(0)Y_{t-i}$, then Eq. (8) can be rewritten as follows:

$$V_t = X_t + \theta_1 X_{t-1} + \cdots + \theta_q X_{t-q} = \sum_{i=0}^{q}\theta_i X_{t-i} \tag{10}$$

According to the time series analysis theory of stationary processes, Eq. (10) can be equivalently converted into an auto-regressive (AR) model by utilizing the lag operator:

$$V_t = \sum_{i=1}^{p}\varphi_i V_{t-i} + X_t \tag{11}$$

where φ_i are the auto-regressive coefficients, and p is the order of the model.

By normalizing the random variables, i.e., $X_t = \sqrt{2\pi\Delta t}\,h(0)\bar{X}_t$, Eq. (11) can be further written as follows:

$$V_t = \sum_{i=1}^{p} \varphi_i V_{t-i} + L\bar{X}_t \tag{12}$$

where $L = \sqrt{2\pi\Delta t h(0)}$ is the gain coefficient, and $\bar{X}_t$ is the orthogonal variable with unit variance, which satisfies the following conditions:

$$E\left[\bar{X}_t\right] = 0,\ \ E\left[\bar{X}_t \bar{X}_{t'}\right] = \delta_{tt'} \tag{13}$$

In Eq. (13), the time series variable V_t and the orthogonal variable $\bar{X}_t$ satisfy the following condition:

$$E\left[V_s \bar{X}_t\right] = 0, \quad s < t \tag{14}$$

3. VAR model of stationary stochastic vector processes

Based on the AR model, the VAR model for vector time series $\boldsymbol{V}_t$ can be expressed as follows [7]:

$$\boldsymbol{V}_t = \sum_{i=1}^{p} \boldsymbol{\varphi}_i \boldsymbol{V}_{t-i} + \boldsymbol{L}\bar{\boldsymbol{X}}_t \tag{15}$$

where $\boldsymbol{\varphi}_i\ (i = 1, 2, \cdots, p)$ and $\boldsymbol{L}$ are the $n \times n$ VAR coefficient matrix and the gain coefficient matrix, respectively; $\bar{\boldsymbol{X}}_t = \left[\bar{X}_{t,1}, \bar{X}_{t,2}, \cdots, \bar{X}_{t,n}\right]^{\mathrm{T}}$ is a standardized vector time series, of which the components, i.e., $\bar{X}_{t,k}\ (k = 1, 2, \cdots, n)$, are orthogonal random variables which satisfy the following conditions:

$$E\left[\bar{X}_{t,k}\right] = 0, \ \ k = 1, 2, \cdots, n \tag{16a}$$

$$E\left[\bar{X}_{s,l}\bar{X}_{t,k}\right] = \delta_{kl}\delta_{st}, \ \ k, l = 1, 2, \cdots, n \tag{16b}$$

Similar to the univariate stochastic process, the vector time series $\boldsymbol{V}_t$ and $\bar{\boldsymbol{X}}_t$ satisfy the following condition:

$$E\left[\boldsymbol{V}_s \bar{\boldsymbol{X}}_t^{\mathrm{T}}\right] = \boldsymbol{0}_{n\times n}, \ \ s < t \tag{17}$$

4. Random functions representation of the orthogonal random variables

To avoid the difficulty of representative point selection for high-dimensional random variables, an intermediate random variable set $\left\{\tilde{X}_{s,l}\right\}\ (s = 1, 2, \cdots N;\ l = 1, 2, \cdots, n)$ is defined as a function of the elementary random vector $\boldsymbol{\Theta}$. Following the dimension-reduction concept of random orthogonal functions [8], the DR-VAR simulation formula is given as follows:

$$\tilde{X}_{s,l} = 2\cos(l \times \Theta_1 + \alpha) \times \sin(s \times \Theta_2 + \beta), \quad s = 1, 2, \cdots, N;\ l = 1, 2, \cdots, n \tag{18}$$

where Θ_1 and Θ_2 denote two independent elementary random variables, following a uniform distribution over the interval $(0, 2\pi) \times (0, 2\pi)$; α and β are deterministic values and are valued by $\pi/4$.

The random variables $\tilde{X}_{s,l}$ need to be mapped to the target orthogonal random variables $\bar{X}_{t,k}$ via a shuffle mapping, i.e., $\tilde{X}_{s,l} \to \bar{X}_{t,k}\ (s, t = 1, 2, \cdots, N;\ l, k = 1, 2, \cdots, n)$. This mapping can be efficiently implemented by the MATLAB functions "rand('state',0)" and "randperm$(n \times N)$".

5. Verification of DR-VAR model

The two-sided APSD is selected as the Simiu spectrum model [9], which is given as follows:

$$S(z,\omega) = 200u_*^2 \frac{f}{\omega(1+50f)^{5/3}} \tag{19a}$$

$$f = \omega z / \left(2\pi\bar{v}(z)\right) \tag{19b}$$

where z denotes the height coordinate of the point; u_* is the friction velocity; $\bar{v}(z)$ is the mean wind speed at a height of z, and the logarithmic velocity profile model is used to calculate the mean wind speed:

$$\bar{v}(z) = \frac{u_*}{\kappa}\ln\left(\frac{z}{z_0}\right) \tag{20}$$

where $\kappa = 0.4$ denotes the Kármán constant; z_0 is the ground roughness length.

The two-dimensional Davenport spatial coherence function model [10] is selected to calculate the CPSD of the wind field, and the corresponding representation is given as follows:

$$\gamma_{ij}(\omega) = \exp\left[\frac{-\omega\sqrt{C_y^2(y_i-y_j)^2 + C_z^2(z_i-z_j)}}{\pi\left[\bar{v}(z_i)+\bar{v}(z_j)\right]}\right] \tag{21}$$

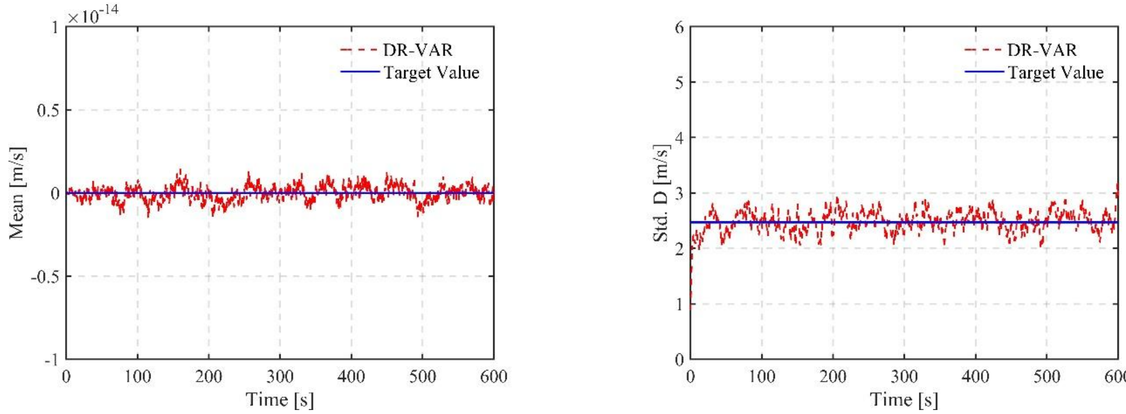
where C_y and C_z are decay coefficients in y, z directions between two spatial points.

The simulation parameters are listed in Table 1.

Table 1 – Parameters for simulating stochastic wind speed vector processes

Parameters	Values
Duration T (s)	600
Time interval Δt (s)	0.25
Ground roughness length z_0 (m)	0.02
Order of the AR model p	100
Friction velocity u_* (m/s)	1.03
Decay coefficients C_y and C_z	10
Number of representative points n_{sel}	523

Fig. 1 compares the mean value and standard deviation of the simulated time series with the target values, respectively. The mean value of the DR-VAR simulated series reaches 10^{-14}, indicating that the wind speed time series generated by the proposed dimension-reduction method is a typical zero-mean stochastic process.



(a) Mean (b) Std. D

Fig. 1 – Comparison of mean values and standard deviations between the simulated wind speed time series and target values.

Fig. 2 compares the simulated APSD with the target spectrum. The simulated APSD agrees well with the target over the entire frequency range of interest, further validating the accuracy of the proposed method.

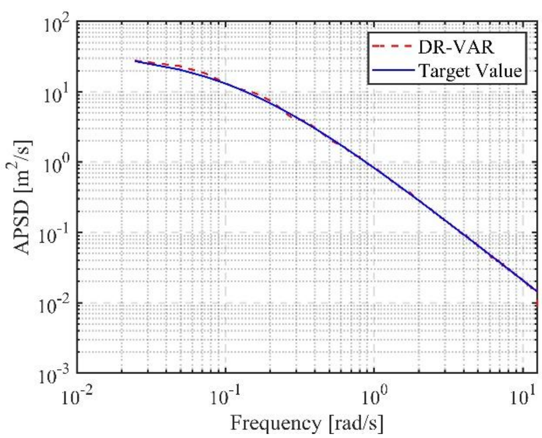


Fig. 2 – Comparisons of APSDs between simulated and target values.

To quantitatively evaluate the accuracy of DR-VAR methods, the average relative errors (ARE) of the mean, standard deviation, and APSD are computed and compared with those of the MC method, as shown in Table 2. From the table one can see that the DR-VAR method consistently achieves higher simulation accuracy than the conventional Monte Carlo-based VAR model for the same number of sample realizations.

Table 2 – Comparison of the average relative errors

Methods	n_{sel}	Mean (%)	Std. D (%)	APSD (%)
Dimension-reduction method	523	1.06×10^{-14}	4.28	1.97
Monte Carlo method	523	3.72	6.41	3.85
	1000	2.30	5.78	3.22

Fig. 3 shows the representative wind velocity time histories at $z = 20m$ generated by the DR-VAR method and the MC method, respectively. The results show that the time series generated exhibit the typical features of stochastic turbulent wind velocity.

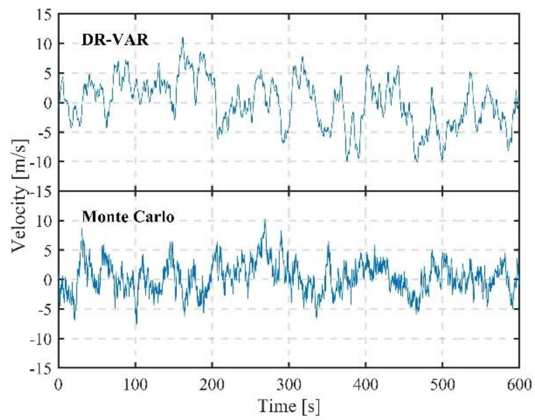


Fig. 3 – Wind velocity time histories at $z = 20m$

6. Conclusion

This study proposes a dimension-reduction vector autoregressive (DR-VAR) method for the simulation of stationary stochastic vector processes. By introducing random orthogonal functions, the high-dimensional random variables in the conventional VAR model are represented by only two elementary random variables, thereby overcoming the dimension dilemma associated with Monte Carlo-based implementations. Numerical results show that the proposed method achieves higher accuracy in the mean value, standard deviation, and APSD than the conventional VAR model with the same sample size, while also exhibiting slightly better computational efficiency. In addition, the representative sample set generated by the DR-VAR method contains complete probabilistic information, making it suitable for rigorous stochastic dynamic response and reliability analyses. Parametric studies indicate that increasing the number of representative points is more effective than increasing the model order in improving simulation accuracy.

7. Acknowledgements

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8. References

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